Theoretical Statistics Qualifying Exam 2025: STA 684

General Instruction: Answer all questions. Print your answer, your name, number the pages and number the problems on provided exam paper. Write on one side only.

1. Let $X_1, \ldots, X_n$ be iid random variables each having a normal distribution with mean μ and variance σ^2. Prove that $\bar{X}$ and S^2 are independent. (12)

2. Let the random variable X_n have a distribution that is $b(n, p)$. Prove that $(1 - X_n/n)$ converges in probability to $(1 - p)$. (12)

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ distribution, where σ^2 is fixed but $-\infty < \theta < \infty$. Show that the mle of θ is $\bar{X}$. (12)

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(\mu_0, \theta\sigma^2)$ distribution, where $0 < \theta < \infty$ and μ_0 is known. Show that the likelihood ratio test of H_0: $\theta = \theta_0$ versus H_1: $\theta \neq \theta_0$ can be based upon the statistic

$$W = \frac{\sum_{i=1}^{n} (X_i - \mu_0)^2}{\theta_0}$$

Determine the null distribution of W and give, explicitly, the rejection rule for a level α test. (12)

5. Let $X_1,\ldots,X_{n1}$ be a random sample from the distribution of $X \sim N(\mu_1, \sigma_1^2)$ and let $Y_1,\ldots,Y_{n2}$ be a random sample from the distribution of $Y \sim N(\mu_2, \sigma_2^2)$. Prove that (12)

$$\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \Big/ \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \xrightarrow{P} 1$$

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution having parameter θ, $0 < \theta < \infty$. Prove that the sum of the observations of the random sample of size n is a sufficient statistic for θ. (10)

7. If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution having pdf of the form $f(x; \theta) = \theta x^{\theta-1}$, $0 < x < 1$, zero elsewhere, show that a best critical region for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$ is $C = \{(x_1, x_2, \ldots, x_n) : c \leq \prod_1^n x_i\}$. (15)

8.
   a. State Neyman–Pearson Theorem. (10)

   b. Prove Neyman–Pearson Theorem. (5)