DEPARTMENT OF MATHEMATICS
PH.D. QUALIFYING EXAMINATION – STATISTICS
January 2023

General Instructions

• There are two parts in this exam: STA 584 and STA 682. You are to answer all questions. The score for each part will be converted to its percentage.

• Write on one side only. Begin each subpart on a new sheet with the problem number noted. You must **show** all your work and **justifications** correctly and completely to receive full credits. Partial credits may be given for partially correct solutions.

• For each problem/subproblem, hand in only the answer that you want to be graded. If necessary, please make clear, e.g., by crossing out the other answer(s), which answer should be graded. Crossed-out work will be ignored. Failure to follow this *instruction* for a *problem will result* in a *zero score* for that *problem*.

• If a theorem is applied, you must clearly state the theorem, identify its assumption(s) and conclusion(s), and justify why it is applicable. New notations must be defined before use.

• When finished, please collate all pages according to the problem numbers and then number the pages accordingly. Hand in also the exam paper.

By signing below, I hereby acknowledge that I have completely read and fully understand the instructions.

____
Signature

____
Printed Name

PART A: STA 584

This part consists of four problems, each with subparts. It has a possible total of 51 points.

Problem 1: **(17 points)** Solve the following problems.

1.a) A reservation service employs three information operators who receive requests for information independently of one another, each according to Poisson Process with a rate of 1 per minute. What is the probability that during a given 1-minute period, no more than one of the three operators receives at least two requests? (4 points)

1.b) There is a 50-50 chance that the queen carries the gene of hemophilia. If she is a carrier, then each prince has a 50-50 chance of having hemophilia independently. If the queen is not a carrier, the prince will not have the disease. Suppose the queen has had two princes without the disease. What is the probability the queen is a carrier? (4 points)

1.c) The probability that a person living in a city owns a dog is 0.3. Find the probability that the sixth person randomly interviewed in the city is the second one to own a dog. (2 points)

1.d) Let Y be a random variable with a moment generating function of $e^{4t}/(4-t^2)$, where $|t| < 2$. Find $E(Y^2)$. (4 points)

1.e) Derive the moment-generating function of the exponential distribution with a mean of θ. (3 points)

Problem 2: (19 points)

Consider two random variables X and Y whose joint probability density function is given by

$$f(x,y) = \begin{cases} 8xy & \text{for } 0 < x < y < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

2.a) Find the probability density function of X. (2 points)
2.b) $P(X > 1/3 \mid X \geq 1/2)$. (2 points)
2.c) Define $U = X - 2Y$. Assume that $E(Y) = 4/5$ and $E(Y^2) = 2/3$. Find the variance of U. (8 points)
2.d) Derive the conditional mean of Y given $X = x$, where $0 < x < 1$. (3 points)
2.e) Define $V = X/Y$. Find the cumulative distribution function of V. (4 points)

Problem 3: (9 points)

Consider two random variables X and Y whose joint probability mass function is given by

		x	
		1	2
y	0	1/10	3/20
	1	2/10	3/10
	2	1/10	3/20

3.a) Define $U = X^2$. Find the moment generating function of U. (3 points)
3.b) Find the conditional mean of U given $Y = 1$. (3 points)
3.c) Are the variables U and Y independent? Justify your answer. (3 points)

Problem 4: (6 points, 1 point/each)

Determine if each of the following statements is true or false. *A*nswer *True* or *False.*

__ 4.a) The exponential and chi-squared distributions are special cases of the gamma distribution.

__ 4.b) A negative binomial random can be expressed as the sum of independent and identical Bernoulli random variables.

__ 4.c) Let X be the number of trials until a success in a sequence of independent and identical Bernoulli trials. Then X has a geometric distribution.

__ 4.d) The sum of correlated normally distributed random variables is not normally distributed.

__ 4.e) Chebyshev's theorem or inequality is derived under the assumption of a mound-shaped distribution.

__ 4.f) Let σ_X, $\sigma_{Y,}$ and σ_{X+Y} be the standard deviations of the random variables X, Y, and $X + Y$, respectively. Then σ_{X+Y} is less than or equal to the sum of σ_X and $\sigma_{Y,}$

PART B: STA 684

This part consists of four problems, each with subparts. It has a possible total of 80 points.

Problem 1: (10 points)

1.a) **(5 points)** If $X_n \xrightarrow{D} X$ and $\frac{Y_n}{X_n} \xrightarrow{P} 0$, then show that $Y_n \xrightarrow{P} 0$.

1.b) **(5 points)** Consider $X_1, \dots, X_n$ to be a random sample such that $E(X_1) = 0$ and $V(X_1) = 1$ and consider the statistic $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ such that Y_n converges in distribution to Y. Find $P(Y > 1)$.

Problem 2: (30 points)

Let $X_1, \dots, X_n$ be a random sample from a population with the following density:

$$f(x) = \frac{1}{\gamma(1+x)^{\frac{1}{\gamma}+1}}; \; x \geq 0; \; \gamma > 0$$

2.a) (5 points) Find $\hat{\gamma}$, the MLE of γ.
2.b) (5 points) Show that $\hat{\gamma}$ is a consistent estimator of γ.
2.c) (10 points) Find the limiting distribution of $\hat{\gamma}$.

We would like to conduct the following test of hypothesis H$_o$: $\gamma = 1$ vs. H$_1$: $\gamma > 1$ at α level of significance. Use the sampling distribution in (2.c) to

2.d) (5 points) obtain the test's rejection region;
2.e) (5 points) obtain the test's power function.

Problem 3: (20 points)

The number of accidents per month at a certain intersection in San Diego is assumed to follow a Poisson distribution with a rate of λ accidents per month. Let $X_1, \dots, X_n$ be the numbers of monthly accidents observed over a period of *n* consecutive months. Due to the almost stable weather conditions of San Diego, the X_i's can be assumed to be independent and identically distributed.

3.a) (5 points) Find a sufficient statistic for the assumed distribution model. Is it complete? Justify your answer.

We'd like to estimate θ, the probability that next month will be accident-free, i.e., $\theta = P(X=0)$. Since the MLE of θ is biased, we consider the unbiased but inefficient estimator:

$$\hat{\theta} = \mathbf{1}(X_1 = 0) = \begin{cases} 1, & X_1 = 0 \\ 0, & Otherwise \end{cases}$$

3.b) (10 points) Use some appropriate technique to obtain an improved estimator of θ out of $\hat{\theta}$.
3.c) (5 points) Is the estimator you obtained in (3.b) an MVUE? Justify your answer.

Problem 4: (20 points)

A random sample of size *n* is selected from a population with pdf:

$$f(x) = \frac{2}{\theta}\left(\frac{x}{\theta}\right);\ 0 \le x \le \theta.$$

4.a) (5 points) Find $\hat{\theta}$, the MLE of θ. Justify your answer.
4.b) (10 points) Find the sampling distribution of the MLE of θ.
4.c) (5 points) Is $\hat{\theta}$ an unbiased estimator of θ. Justify your answer.