STA 684 Part

1. (10 *points*) Suppose $X_1, X_2, \dots, X_n$ constitute a random sample from a normal population with mean θ and variance θ^2. Let $\bar{X}$ and S are, respectively, the sample mean and sample standard deviation, and let $c = \sqrt{\frac{n-1}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$. Which of the following is/are true? You must justify your answer.

   (1.a) (3 *points*) For any constant b the estimator $b\bar{X} + (1-b)cS$ is an unbiased estimator for θ.

   (1.b) (4 *points*) The estimator given in Part (1.a) has minimum variance when $b = 1 - 1/\{1 + n(c^2 - 1)\}$.

   (1.c) (3 *points*) The $(\bar{X}, S^2)$ is a sufficient statistic for θ.

2. (10 *points*) Consider a random sample $X_1, X_2, \dots, X_n$ from the pdf $f(x; \mu) = \sqrt{1/(2\pi x^3)}\, e^{-\frac{1}{2x}(-1+x/\mu)^2}, 0 < x < \infty, \mu > 0.$

   (2.a) (5 *points*) Prove that the most powerful critical region of size α to test $H_0: \mu = 1$ vs $H_a: \mu = 2$ for some constant k is $\bar{x} \geq k$. Explain how to find the constant k exactly. [Hint: use the first formula of the moment generating function given in Problem 4].

   (2.b) (5 *points*) Prove that the critical region of size α to test $H_0: \mu = 1$ vs $H_a: \mu \neq 1$ for some constant k is $\bar{x} + 1/\bar{x} \geq k$. Explain how to find the constant k approximately.

3. (30 *points*) Let $X_1, X_2, \dots, X_n$ constitute a random sample from the following density function.

$$f(x) = \begin{cases} \sqrt{\dfrac{\alpha^2\theta}{2\pi x^{\alpha+2}}}\, e^{-\frac{\theta}{2x^\alpha}\left(\frac{x^\alpha-\mu}{\mu}\right)^2} & \text{for } x > 0, \mu > 0, \sigma > 0 \\ 0 & \text{otherwise,} \end{cases}$$

   where α is a known positive constant.

   The moment generating functions of X^α and $X^{-\alpha}$ are respectively given by

$$M_{X^\alpha}(t) = e^{\frac{\theta}{\mu}\left(1-\sqrt{1-\frac{2t\mu^2}{\theta}}\right)}, t < \frac{\theta}{2\mu^2} \text{ and } M_{X^{-\alpha}}(t) = e^{\frac{\theta}{\mu}\left(1-\sqrt{1-\frac{2t}{\theta}}\right)} \Big/ \sqrt{1-\frac{2t}{\theta}}, t < \frac{\theta}{2}.$$

   (4.a) (5 *points*) Find the maximum likelihood estimators, $\hat{\mu}$ and $\hat{\theta}$ for the parameters μ and θ.

(4.b) (5 *points*) Obtain the distribution of $\hat{\mu}$ and hence show that $\hat{\mu}$ is a (minimal) sufficient statistic for μ. Is the estimator $\hat{\mu}$, consistent? Justify your answer.

(4.c) (5 *points*) Find the Rao-Cramér lower bound for the estimator $\hat{\mu}$. Is the estimator $\hat{\mu}$, efficient? Justify your answer.

(4.d) (5 *points*) Derive the formula for expected (Fisher) information matrix, $I(\mu, \theta)$.

(4.e) (5 *points*) Find the maximum likelihood estimator for the variance of $X^{-\alpha}$.

(4.f) (5 *points*) Using the Delta method, show that the variance of the estimator obtained in Part (4.e) is given by

$$\frac{1}{n\hat{\mu}^2\hat{\theta}^4}\left(32\hat{\mu}^2 + 17\hat{\mu}\hat{\theta} + 2\hat{\theta}^2\right).$$